

ON THE BIRCH–SWINNERTON-DYER QUOTIENTS MODULO SQUARES

TIM AND VLADIMIR DOKCHITSER

to John Coates

ABSTRACT. Let A be an abelian variety over a number field K . An identity between the L -functions $L(A/K_i, s)$ for extensions K_i of K induces a conjectural relation between the Birch–Swinnerton-Dyer quotients. We prove these relations modulo finiteness of III, and give an analogous statement for Selmer groups. Based on this, we develop a method for determining the parity of various combinations of ranks of A over extensions of K . As one of the applications, we establish the parity conjecture for elliptic curves assuming finiteness of $\mathrm{III}(E/K(E[2]))[6^\infty]$ and some restrictions on the reduction at primes above 2 and 3: the parity of the Mordell-Weil rank of E/K agrees with the parity of the analytic rank, as determined by the root number. We also prove the p -parity conjecture for all elliptic curves over \mathbb{Q} and all primes p : the parities of the p^∞ -Selmer rank and the analytic rank agree.

CONTENTS

1. Introduction	2
2. \square -Conjecture and regulator quotients	5
2.1. Artin formalism and BSD-quotients	5
2.2. \square -Conjecture	6
2.3. Regulator quotients and ranks	8
2.4. Regulator constants: examples	11
3. Tamagawa numbers and root numbers for elliptic curves	12
3.1. Review of root numbers	12
3.2. The case of $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ -extensions	13
3.3. Application to the Parity Conjecture	15
4. Selmer Groups	16
4.1. Invariance of the BSD-quotient for Selmer groups	17
4.2. Isogenies between products of Weil restrictions	19
4.3. Determining Q	19
4.4. Example: Selmer ranks for $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ -extensions	22
4.5. Example: Dihedral groups	25
4.6. Application to the p -Parity Conjecture over \mathbb{Q}	26
References	27

2000 *Mathematics Subject Classification*: Primary 11G40; Secondary 11G05, 11G07, 11G10, 14G25

1. INTRODUCTION

The celebrated conjecture of Birch, Swinnerton-Dyer and Tate asserts that for every elliptic curve E over a number field K , its Mordell-Weil rank coincides with the order of vanishing of its L -function at $s = 1$. The parity of the latter is determined by the root number $w(E/K) = \pm 1$, the sign in the expected functional equation of the L -function, leading to

Conjecture 1.1 (Parity Conjecture). *The Mordell-Weil rank $\mathrm{rk}(E/K)$ is even if and only if the root number $w(E/K)$ is $+1$.*

Save for the rank 0 and 1 cases over \mathbb{Q} , virtually nothing is known about this problem. At best, one can only lay hands on the p^∞ -Selmer rank $\mathrm{rk}_p(E/K)$ for a prime p , that is the Mordell-Weil rank plus the number of copies of $\mathbb{Q}_p/\mathbb{Z}_p$ in the Tate-Shafarevich group $\mathrm{III}(E/K)$. The Parity Conjecture can also be formulated for Selmer ranks, as III is supposed to be finite by the Shafarevich-Tate conjecture:

Conjecture 1.2 (p -parity). *$\mathrm{rk}_p(E/K)$ is even if and only if $w(E/K) = 1$.*

In view of the conjectures, the definition of the root number as a product of local terms (local root numbers) suggests that the parities of $\mathrm{rk}(E/K)$ and $\mathrm{rk}_p(E/K)$ should be governed by local data of the elliptic curve. The purpose of the paper is to develop a theory that provides such a “local-to-global” expression for various combinations of ranks of E over extensions of K . The exact description of these “computable” combinations is a curious group-theoretic problem that we have not addressed. However, there are enough of them to enable us to prove:

Theorem 1.3. *Assuming the Shafarevich-Tate conjecture, Conjecture 1.1 holds over all number fields for elliptic curves with semistable reduction at primes $v|6$ and not supersingular at $v|2$.*

Theorem 1.4. *Conjecture 1.2 holds for all E/\mathbb{Q} and all primes p .*

Our starting point is a conjectural formula implied by Artin formalism for L -functions and the Birch-Swinnerton-Dyer conjecture. As above, fix an elliptic curve E/K (or a principally polarised abelian variety). Suppose L_i, L'_j are finite extensions of K such that the $\mathrm{Gal}(\bar{K}/K)$ -representations $\bigoplus_i \mathrm{Ind}_{L_i/K} \mathbf{1}_{L_i}$ and $\bigoplus_j \mathrm{Ind}_{L'_j/K} \mathbf{1}_{L'_j}$ are isomorphic. Then

$$\prod_i L(E/L_i, s) = \prod_j L(E/L'_j, s),$$

by Artin formalism. Ignoring rational squares, the conjectural expression for the leading terms at $s = 1$ leads to a relation between the regulators and Tamagawa numbers that we will refer to as the \square -Conjecture. For instance, for semistable elliptic curves it reads

$$\prod_i \mathrm{Reg}(E/L_i) c(E/L_i) \equiv \prod_j \mathrm{Reg}(E/L'_j) c(E/L'_j) \pmod{\mathbb{Q}^{*2}},$$

with c the product of local Tamagawa numbers. We will show that the \square -Conjecture follows from the Shafarevich-Tate conjecture.

The crucial observation is that the regulators need not cancel by themselves. It turns out that their quotient can always be expressed through a combination of Mordell-Weil ranks, whose parity is therefore determined by local data. Here is an illustration of how this works in the simplest possible setting, semistable elliptic curves in S_3 -extensions:

Example 1.5. Suppose $\text{Gal}(F/K) \cong S_3$, and let M, L be intermediate extensions of degrees 2 and 3 over K , respectively. There is a relation

$$(\text{Ind}_{F/K} \mathbf{1}_F) \oplus \mathbf{1}_K^{\oplus 2} \cong (\text{Ind}_{M/K} \mathbf{1}_M) \oplus (\text{Ind}_{L/K} \mathbf{1}_L)^{\oplus 2}.$$

(i) For semistable E/K , the \square -Conjecture implies that

$$\frac{\text{Reg}(E/F) \text{Reg}(E/K)^2}{\text{Reg}(E/M) \text{Reg}(E/L)^2} \equiv \frac{c(E/F)c(E/K)^2}{c(E/M)c(E/L)^2} \pmod{\mathbb{Q}^{*2}}.$$

(ii) The quotient of regulators is related to Mordell-Weil ranks (Ex. 2.18):

$$3^{\text{rk}(E/K) + \text{rk}(E/M) + \text{rk}(E/L)} \equiv \frac{\text{Reg}(E/F) \text{Reg}(E/K)^2}{\text{Reg}(E/M) \text{Reg}(E/L)^2} \pmod{\mathbb{Q}^{*2}}.$$

Thus, assuming finiteness of III, we obtain an expression for the sum of the three ranks $\text{rk}(E/K) + \text{rk}(E/M) + \text{rk}(E/L)$ in terms of local data.

(ii') In fact, by a somewhat more sophisticated technique, we can prove an analogous (unconjectural) statement about 3^∞ -Selmer ranks (Thm. 4.11):

$$\text{rk}_3(E/K) + \text{rk}_3(E/M) + \text{rk}_3(E/L) \equiv \text{ord}_3 \frac{c(E/F)c(E/K)^2}{c(E/M)c(E/L)^2} \pmod{2}.$$

(iii) Finally, a purely local computation allows us to relate the Tamagawa numbers to root numbers (Prop. 3.3):

$$w(E/K)w(E/M)w(E/L) = 1 \iff \text{ord}_3 \frac{c(E/F)c(E/K)^2}{c(E/M)c(E/L)^2} \equiv 0 \pmod{2},$$

and we obtain a special case of the parity conjecture for S_3 -extensions.

The layout of the paper is as follows:

In §§2.1–2.2 we formulate the \square -Conjecture and prove it assuming finiteness of III (Conj. 2.4, Cor. 2.5). This relies on invariance of the BSD-quotient under Weil restriction of scalars and under isogenies. Next, we relate the quotient of regulators from the conjecture to the parity of Mordell-Weil ranks in §2.3 (Thm. 2.12, Cor. 2.13), and give examples in §2.4.

Thus, we have now complete versions of steps (i) and (ii) of the above example (principally polarised abelian varieties and arbitrary field extensions). We do not attempt to deal with (iii) in such generality, but confine ourselves to elliptic curves and extensions with Galois group $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_p)$. After reviewing the classification of root numbers in §3.1, we relate the Tamagawa numbers to root numbers for such extensions in §3.2 (Prop. 3.3). Combined with the results of [11] on the parity conjecture for elliptic curves with a 2-isogeny, this proves Theorem 1.3.

So far, we related parities of Mordell–Weil ranks to Tamagawa numbers assuming that III is finite. In §4 we address the problem of getting an unconditional statement about Selmer ranks (as in (ii')). We prove an analogue of the \square -Conjecture (Thm. 4.3, Cor. 4.5) by tweaking Tate–Milne’s proof of the isogeny invariance of the Birch–Swinnerton–Dyer Conjecture. The quotient of regulators becomes replaced by a quantity Q measuring the effect of an isogeny on Selmer groups. In §4.3 we turn Q into Selmer ranks in fair generality (Thm. 4.7, Cor. 4.8), and we illustrate it for S_n -extensions (Ex. 4.9), $\binom{1}{0}^*$ -extensions (§4.4), and dihedral extensions (§4.5). In §4.4 we give an application to ranks of elliptic curves in false Tate curve towers. We end in §4.6 by proving Theorem 1.4; for odd p it is a consequence of our results for dihedral extensions and the existence of quadratic and anticyclotomic twists for which the Birch–Swinnerton–Dyer rank formula is known to hold.¹

Finally, let us mention how the applications of our theory connect to earlier work. To our best knowledge, over number fields Theorem 1.3 is the first general result of this kind, except for the work [7, 11] on curves with a p -isogeny. In contrast, the p -parity conjecture over \mathbb{Q} was known in almost all cases, thanks to Birch, Stephens, Greenberg and Guo [3, 15, 16] (E CM), Kramer, Monsky [21, 26] ($p = 2$), Nekovář [28] (p potentially ordinary or potentially multiplicative) and Kim [18] (p supersingular). The results for Selmer groups in dihedral and false Tate curve extensions are similar to those recently obtained by Mazur–Rubin [23] and Coates–Fukaya–Kato–Sujatha [7, 8], respectively.

Notation. Throughout the paper K always denotes a number field. For a place v of K we write $|\cdot|_v$ for the normalised absolute value at v . If L/K is a finite extension, we denote by $\text{Ind}_{L/K}\mathbf{1}_L$ the induction of the trivial (complex) representation of $\text{Gal}(\bar{K}/L)$ to $\text{Gal}(\bar{K}/K)$. This is the permutation representation corresponding to the set of K -embeddings of L into \bar{K} .

For an elliptic curve E/K we use the following notation:

$\text{rk}(E/K)$	Mordell–Weil rank of E/K .
$\text{rk}_p(E/K)$	p^∞ -Selmer rank of E/K , i.e.
	$\text{rk}(E/K) +$ number of copies of $\mathbb{Q}_p/\mathbb{Z}_p$ in $\text{III}(E/K)$.
$w(E/K_v)$	local root number of E at a place v of K .
$w(E/K)$	global root number, $= \prod_v w(E/K_v)$.
$\text{Reg}(E/K)$	regulator of E/K , i.e. $ \det $ of the canonical height pairing on a basis of $E(K)/E(K)_{\text{tors}}$.
c_v	local Tamagawa number at a finite place v .
$c(E/K)$	product of the local Tamagawa numbers, $= \prod_{v \nmid \infty} c_v$.
$W_{F/K}(E)$	the Weil restriction of scalars of E/F to K .

Finally, we will need a slight modification of $c(E/K)$. Fix an invariant differential ω on E . Let ω_v^o be Néron differentials at finite places v of K ,

¹Since writing of this paper, we have extended (ii') and (iii) to arbitrary $\text{Gal}(F/K)$ in [12, 13]; for (ii') the theory is now as clean as for (ii), e.g. computations with isogenies in §§4.4–4.5 are replaced by elementary representation theory, as in §2.4.

and set

$$C(E/K) = \prod_{v \nmid \infty} c_v \left| \frac{\omega}{\omega_v^o} \right|_v.$$

Note that $C(E/K)$ depends on the choice of ω , although we have omitted this from the notation. When writing $C(E/L_i)$ for various extensions L_i/K , we always implicitly use the same K -rational differential.

We use similar notation for abelian varieties (the analogue of an invariant differential being a non-zero global exterior form). We write A^t for the dual abelian variety.

Acknowledgements. Parts of this research have been carried out while the first author (T.) was a University Research Fellowship holder of the Royal Society and while the second author (V.) stayed at the Max Planck Institute for Mathematics (Bonn). We are grateful to both institutions for their support. We would also like to thank Henri Darmon, Claus Diem, Matteo Longo and Robert Prince for helpful discussions, and the referee for the comments on the paper.

2. \square -CONJECTURE AND REGULATOR QUOTIENTS

2.1. Artin formalism and BSD-quotients. Let K be a number field and let A/K be an abelian variety with a fixed global exterior form ω . Recall the statement of the Birch–Swinnerton-Dyer conjecture:

Conjecture 2.1 (Birch–Swinnerton-Dyer, Tate [35]).

- (1) *The L -function $L(A/K, s)$ has an analytic continuation to $s = 1$, and*

$$\text{ord}_{s=1} L(A/K, s) = \text{rk}(A/K).$$
- (2) *The Tate-Shafarevich group $\text{III}(A/K)$ is finite, and the leading coefficient of $L(A/K, s)$ at $s = 1$ is*

$$\text{BSD}(A/K) = \frac{|\text{III}(A/K)| \text{Reg}(A/K) C(A/K)}{|A(K)_{\text{tors}}| |A^t(K)_{\text{tors}}| |\Delta_K|^{\dim A/2}} \prod_{\substack{v \mid \infty \\ \text{real } A(K_v)}} \int |\omega| \prod_{\substack{v \mid \infty \\ \text{cplx } A(K_v)}} 2 \int \omega \wedge \bar{\omega}.$$

Notation. We call $\text{BSD}(A/K)$ the Birch–Swinnerton-Dyer quotient for A/K . We also write $\text{BSD}_p(A/K)$ for the same expression with III replaced by its p -primary component $\text{III}[p^\infty]$. (They are independent of the choice of ω by the product formula.)

Now let $L_i \supset K$ and $L'_j \supset K$ be number fields such that

$$\bigoplus_i \text{Ind}_{L_i/K} \mathbf{1}_{L_i} \cong \bigoplus_j \text{Ind}_{L'_j/K} \mathbf{1}_{L'_j}$$

as complex representations of $\text{Gal}(\bar{K}/K)$. In other words, the $\text{Gal}(\bar{K}/K)$ -sets $\coprod_i \text{Hom}_K(L_i, \bar{K})$ and $\coprod_j \text{Hom}_K(L'_j, \bar{K})$ give rise to isomorphic permutation representations. By Artin formalism for L -functions,

$$\prod_i L(A/L_i, s) = \prod_j L(A/L'_j, s),$$

so the following is a consequence of Conjecture 2.1.

Conjecture 2.2. *With A/K and L_i, L'_j as above,*

- (a) $\sum_i \text{rk}(A/L_i) = \sum_j \text{rk}(A/L'_j)$,
- (b) $\text{III}(A/L_i), \text{III}(A/L'_j)$ are finite, and $\prod_i \text{BSD}(A/L_i) = \prod_j \text{BSD}(A/L'_j)$.

This is in effect a compatibility statement of the Birch–Swinnerton-Dyer conjecture with Artin formalism. Part (a) is easily seen to be true: let F/K be a finite Galois extension containing L_i and L'_j , and let $V = A(F) \otimes_{\mathbb{Z}} \mathbb{C}$. Then

$$\text{rk}(A/L_i) = \dim V^{\text{Gal}(F/L_i)} = \langle \mathbf{1}_{L_i}, \text{Res}_{F/L_i} V \rangle = \langle \text{Ind}_{F/L_i} \mathbf{1}_{L_i}, V \rangle$$

by Frobenius reciprocity, and similarly for L'_j ; now take the sum over i and j .

We now show that (b) is implied by finiteness of III. As C. S. Dalawat, K. Rubin and M. Shuter pointed out to us, this is essentially the same as H. Yu’s Theorem 5 in [38].

Theorem 2.3. *Let A/K be an abelian variety, and let L_i, L'_j be finite extensions of K satisfying $\bigoplus_i \text{Ind}_{L_i/K} \mathbf{1}_{L_i} \cong \bigoplus_j \text{Ind}_{L'_j/K} \mathbf{1}_{L'_j}$. Suppose that $\text{III}(A/L_i), \text{III}(A/L'_j)$ are finite. Then Conjecture 2.2b holds.*

Furthermore, if we weaken the assumption to $\text{III}(A/L_i)[p^\infty], \text{III}(A/L'_j)[p^\infty]$ being finite for some prime p , then the p -part of Conjecture 2.2b holds, i.e.

$$\left(\prod_i \text{BSD}_p(A/L_i) \right) / \left(\prod_j \text{BSD}_p(A/L'_j) \right)$$

is a rational number with trivial p -valuation.

Proof. For F/K finite, write $W_{F/K}(A)$ for the Weil restriction of scalars of A/F to K . This is an abelian variety over K of dimension $[F:K] \dim A$, and $\text{BSD}_p(W_{F/K}(A)) = \text{BSD}_p(A/F)$ provided that $\text{III}(A/F)[p^\infty]$ is finite ([24] §1). Consider

$$X = \prod_i W_{L_i/K}(A), \quad Y = \prod_j W_{L'_j/K}(A).$$

Then $\text{BSD}_p(X) = \prod_i \text{BSD}_p(A/L_i)$ and $\text{BSD}_p(Y) = \prod_j \text{BSD}_p(A/L'_j)$. By the invariance of the Birch–Swinnerton-Dyer quotient under isogenies ([6], [35] and [25] Thm. 7.3, Remark 7.4), it suffices to show that X and Y are isogenous.

As the representations $\bigoplus_i \text{Ind}_{L_i/K} \mathbf{1}_{L_i}$ and $\bigoplus_j \text{Ind}_{L'_j/K} \mathbf{1}_{L'_j}$ are realisable over \mathbb{Q} and are isomorphic over \mathbb{C} , they are isomorphic over \mathbb{Q} (see e.g. [32], Ch. 12, Prop. 33 and remark following it). So the corresponding integral permutation modules are isogenous, in the sense that there is an inclusion of one as a finite index submodule of the other. This induces an isogeny $X \rightarrow Y$ (see [24], §2, Prop. 6a). \square

2.2. \square -Conjecture. Although Conjecture 2.2b has the advantage that it does not involve L -functions, it still relies on finiteness of III. Also, even when III is finite it is hard to determine, which makes the statement difficult to work with. However, if A is principally polarised and III is finite, then

the order of III is either a square or twice a square by the non-degeneracy of the Cassels–Tate pairing [34]. (If A is an elliptic curve or has a principal polarisation arising from a K -rational divisor, then the order of III is a square, see [5, 34, 30].) So we can eliminate III from the statement by working modulo squares, which also removes the contribution from the torsion. Moreover, in this combination of BSD-quotients, the discriminants of fields cancel by the conductor-discriminant formula, as do the real and complex periods, provided that one chooses the same ω over K for each term. Thus Conjecture 2.2b implies the following (see Remark 2.7 for an extension to abelian varieties):

Conjecture 2.4 (\square -Conjecture). *Let E/K be an elliptic curve, and fix an invariant differential ω on E . Let L_i, L'_j be finite extensions of K satisfying $\oplus_i \text{Ind}_{L_i/K} \mathbf{1}_{L_i} \cong \oplus_j \text{Ind}_{L'_j/K} \mathbf{1}_{L'_j}$. Then*

$$\prod_i \text{Reg}(E/L_i) C(E/L_i) \equiv \prod_j \text{Reg}(E/L'_j) C(E/L'_j) \pmod{\mathbb{Q}^{*2}}.$$

Corollary 2.5 (of Theorem 2.3). *The p -part of Conjecture 2.4 holds, provided that $\text{III}(E/L_i)[p^\infty]$ and $\text{III}(E/L'_j)[p^\infty]$ are finite. In other words,*

$$\prod_i \text{Reg}(E/L_i) C(E/L_i) / \prod_j \text{Reg}(E/L'_j) C(E/L'_j)$$

is a rational number with even p -valuation.

We are going to explore the (surprisingly non-trivial) consequences of this for parities of Mordell–Weil ranks. Here is a simple example:

Example 2.6. Take the modular curve $E = X_1(11)$ over the fields $\mathbb{Q}, \mathbb{Q}(\mu_3)$, $L = \mathbb{Q}(\sqrt[3]{m})$ and $F = \mathbb{Q}(\mu_3, \sqrt[3]{m})$ for $m > 1$ cube free. We have an equality of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -representations,

$$(\text{Ind}_{F/\mathbb{Q}} \mathbf{1}_F) \oplus (\mathbf{1}_{\mathbb{Q}})^{\oplus 2} \cong (\text{Ind}_{L/\mathbb{Q}} \mathbf{1}_L)^{\oplus 2} \oplus (\text{Ind}_{\mathbb{Q}(\mu_3)/\mathbb{Q}} \mathbf{1}_{\mathbb{Q}(\mu_3)}).$$

The Mordell–Weil rank of E is 0 over $\mathbb{Q}(\mu_3)$, so $\text{Reg}(E/\mathbb{Q})$ and $\text{Reg}(E/\mathbb{Q}(\mu_3))$ are both 1. The \square -Conjecture implies that

$$\frac{\text{Reg}(E/F)}{\text{Reg}(E/L)^2} \equiv \frac{c(E/\mathbb{Q}(\mu_3)) c(E/L)^2}{c(E/F) c(E/\mathbb{Q})^2} \pmod{\mathbb{Q}^{*2}}.$$

For $v|11$, the local Tamagawa number c_v is the valuation of the minimal discriminant ($= -11$) at v , because E has split multiplicative reduction at v . So, by a simple computation, the above quotient of Tamagawa numbers is 1 when $11 \nmid m$ and 3 when $11|m$. On the other hand, let P_1, \dots, P_n be a basis for $E(L) \otimes \mathbb{Q}$, and H the height matrix $\langle P_i, P_j \rangle_L$, so that $\text{Reg}(E/L)$ is $|\det(H)|$ up to a (rational) square. If $g \in \text{Gal}(F/\mathbb{Q})$ is an element of order 3, then $P_1, \dots, P_n, P_1^g, \dots, P_n^g$ is a basis for $E(F) \otimes \mathbb{Q}$. One readily verifies that the height matrix over F is $\begin{pmatrix} 2H & -H \\ -H & 2H \end{pmatrix}$, so the regulator $\text{Reg}(E/F)$ is $3^n |\det(H)|^2$ up to a square. Hence the \square -Conjecture implies that $\text{rk}(E/\mathbb{Q}(\sqrt[3]{m}))$ is odd if and only if $11|m$. (See Example 2.18 and Corollary 2.21 for a generalisation.)

We end with a few observations:

Remark 2.7. There are obvious analogues of the \square -Conjecture and Corollary 2.5 for principally polarised abelian varieties. The only difference is that for $p = 2$ one needs the polarisation to come from a K -rational divisor.

Remark 2.8. For elliptic curves, the local terms c_v and $|\omega/\omega_v^o|_v$ can be obtained from Tate's algorithm, so the conjecture gives an explicit relation between regulators. Note also that the advantage of working with regulators up to rational squares is that one may compute the height matrix on an arbitrary \mathbb{Q} -basis of $E(k) \otimes \mathbb{Q}$.

Remark 2.9. If E/K is semistable, then $C(E/k)$ may be replaced by just the product of the local Tamagawa numbers $c(E/k)$ in 2.2-2.5. Indeed, it suffices to show that above a given prime v of K , the contribution from the differential to $\prod_i C(E/L_i)/\prod_j C(E/L'_j)$ is trivial. But this contribution is easily seen to be the same for every choice of a local differential w_v/K_v , and it is 1 if w_v is minimal (as it stays minimal in every extension).

Remark 2.10. In 2.3 and 2.5, the assumption that $\text{III}[p^\infty]$ is finite for A over all L_i, L'_j follows from its finiteness over their compositum: if A/K is an abelian variety and L/K a finite extension with $\text{III}(A/L)[p^\infty]$ finite, then $\text{III}(A/K)[p^\infty]$ is also finite. Indeed, the Weil restriction of scalars $W_{L/K}(A)$ after an isogeny contains A as a direct summand. Since, by assumption, $\text{III}(A/L)[p^\infty] \cong \text{III}(W_{L/K}(A)/K)[p^\infty]$ is finite, so is $\text{III}(A/K)[p^\infty]$.

2.3. Regulator quotients and ranks. We now explain how to turn the regulator quotients from the \square -Conjecture into parities of Mordell-Weil ranks. If A/K is an abelian variety and $\text{Gal}(F/K) \cong G$, consider the decomposition $A(F) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus \rho_k^{\oplus n_k}$ into \mathbb{Q} -irreducible rational G -representations. We will show that for given $L_i, L'_j \subset F$ the regulator quotient equals $\prod_k \mathcal{C}(\rho_k)^{n_k}$ for purely representation-theoretic quantities $\mathcal{C}(\rho_k)$ (regulator constants) that do not depend on A or the height pairing.

Let G be a finite group, and \mathcal{H} a set of representatives of the subgroups of G up to conjugacy. Its elements are in one-to-one correspondence with transitive G -sets via $H \mapsto G/H$. We call an element of $\mathbb{Z}\mathcal{H}$,

$$\Theta = \sum_i H_i - \sum_j H'_j \quad (H_i, H'_j \in \mathcal{H})$$

a relation between permutation representations if $\bigoplus_i \mathbb{C}[G/H_i] \cong \bigoplus_j \mathbb{C}[G/H'_j]$.

If $\text{Gal}(F/K) \cong G$, then in terms of the fixed fields $L_i = F^{H_i}$ and $L'_j = F^{H'_j}$,

$$\bigoplus_i \text{Ind}_{L_i/K} \mathbf{1}_{L_i} \cong \bigoplus_j \text{Ind}_{L'_j/K} \mathbf{1}_{L'_j}.$$

Notation. Suppose V is a complex representation of G , given with a G -invariant non-degenerate Hermitian inner product \langle , \rangle and a basis $\{e_i\}$. We write $\det \langle , \rangle$ or $\det(\langle , \rangle|V)$ for the determinant of the matrix $(\langle e_i, e_j \rangle)_{ij}$. If V is defined over \mathbb{Q} , then the class of $\det \langle , \rangle$ in $\mathbb{R}^*/\mathbb{Q}^{*2}$ does not depend on the choice of a rational basis.

Definition 2.11. For each \mathbb{Q} -irreducible rational representation ρ of G fix a G -invariant real-valued symmetric positive definite inner product \langle , \rangle on it, and define the *regulator constant*

$$\mathcal{C}(\Theta, \rho) = \frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle | \rho^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle | \rho^{H'_j})} \in \mathbb{Q}^*/\mathbb{Q}^{*2}.$$

It follows from the theorem below that this is independent of the choice of the inner product. (In particular, $\mathcal{C}(\Theta, \rho)$ is indeed in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ rather than $\mathbb{R}^*/\mathbb{Q}^{*2}$, as we can choose \langle , \rangle to be \mathbb{Q} -valued.)

Theorem 2.12. *For any $V \cong \bigoplus_k \rho_k^{n_k}$ with ρ_k rational \mathbb{Q} -irreducible representations,*

$$\frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle | V^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle | V^{H'_j})} = \prod_k \mathcal{C}(\Theta, \rho_k)^{n_k} \pmod{\mathbb{Q}^{*2}},$$

for any G -invariant real-valued symmetric positive definite inner product \langle , \rangle on V .

Corollary 2.13. *Let A/K be a principally polarised abelian variety, and let Θ and $F/K, L_i, L'_i$ be as above. Let $\{\rho_k\}_k$ be the set of \mathbb{Q} -irreducible rational representations of G , and let n_k be the multiplicity of ρ_k in $A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then*

$$\frac{\prod_i \text{Reg}(A/L_i)}{\prod_j \text{Reg}(A/L'_j)} = \prod_k \mathcal{C}(\Theta, \rho_k)^{n_k} \pmod{\mathbb{Q}^{*2}}.$$

In the remainder of §2.3 we prove Theorem 2.12. It suffices to show that the left-hand side is independent of the choice of an inner product.

Lemma 2.14. *Let V be a (complex) vector space and $\langle , \rangle_1, \langle , \rangle_2$ Hermitian inner products. Then $\det \langle , \rangle_1 / \det \langle , \rangle_2$ is independent of the choice of a basis of V .*

Proof. Changing the basis converts the matrix X of an inner product to $M^t X \bar{M}$, where M is the matrix of the basis. The assertion follows from taking the quotient of the determinants. \square

Lemma 2.15. *Let $\Theta = \sum_i H_i - \sum_j H'_j$ be a relation between permutation representations and ρ a complex representation. Then*

$$\sum_i \dim \rho^{H_i} - \sum_j \dim \rho^{H'_j} = 0.$$

Proof. Writing \langle , \rangle_G for the usual inner product on the space of characters,

$$\sum \dim \rho^{H_i} = \sum \langle \text{Res}_{H_i} \rho, \mathbf{1}_{H_i} \rangle_{H_i} = \sum \langle \rho, \text{Ind}^G \mathbf{1}_{H_i} \rangle_G = \langle \rho, \bigoplus \text{Ind}^G \mathbf{1}_{H_i} \rangle_G.$$

There is a similar expression for H'_j and the right-hand sides of the two are the same. \square

Lemma 2.16. *Let $\Theta = \sum_i H_i - \sum_j H'_j$ be as above, and τ a complex irreducible representation with a Hermitian G -invariant inner product. For each subgroup H fix a basis of τ^H and let M_H be the matrix of the inner product on this basis. Suppose $\rho \cong \tau^n$ with some Hermitian G -invariant inner product \langle , \rangle . With respect to the bases of ρ^H induced from those of τ^H by this isomorphism,*

$$\frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle | \rho^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle | \rho^{H'_j})} = \left(\frac{\prod_i \det \frac{1}{|H_i|} M_{H_i}}{\prod_j \det \frac{1}{|H'_j|} M_{H'_j}} \right)^n.$$

In particular the expression is independent of the choice of \langle , \rangle .

Proof. Since the Hermitian G -invariant inner product on τ is unique up to a scalar, the matrix of \langle , \rangle on ρ^H with respect to the induced basis is

$$\begin{pmatrix} \lambda_{11}M_H & \lambda_{12}M_H & \dots & \lambda_{1n}M_H \\ \lambda_{21}M_H & \lambda_{22}M_H & \dots & \lambda_{2n}M_H \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1}M_H & \lambda_{n2}M_H & \dots & \lambda_{nn}M_H \end{pmatrix},$$

for some $n \times n$ matrix $\Lambda = (\lambda_{xy})$ not depending on H . Hence

$$\det(\frac{1}{|H|} \langle , \rangle | \rho^H) = (\det \Lambda)^{\dim \tau^H} (\det \frac{1}{|H|} M_H)^n.$$

The dimensions $\dim \tau^H$ cancel in Θ by Lemma 2.15, and the result follows. \square

Theorem 2.17. *Let $\Theta = \sum_i H_i - \sum_j H'_j$ be as above, and ρ a complex representation of G . Suppose $\langle , \rangle_1, \langle , \rangle_2$ are two Hermitian G -invariant inner products on ρ . For each subgroup H fix a basis of ρ^H . Then, computing with respect to these bases,*

$$\frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle_1 | \rho^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle_1 | \rho^{H'_j})} = \frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle_2 | \rho^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle_2 | \rho^{H'_j})}.$$

Proof. For each subgroup H and each isotypical component $\rho_l \cong \tau_l^{n_l}$ of ρ , choose a basis of τ_l^H and induce a basis of ρ_l^H as in the previous lemma, so

$$\frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle_1 | \rho_l^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle_1 | \rho_l^{H'_j})} = \frac{\prod_i \det(\frac{1}{|H_i|} \langle , \rangle_2 | \rho_l^{H_i})}{\prod_j \det(\frac{1}{|H'_j|} \langle , \rangle_2 | \rho_l^{H'_j})}.$$

The isotypical components of ρ are pairwise orthogonal, so taking direct sums gives the same formula with ρ in place of ρ_l . Finally, applying Lemma 2.14 for every H_i, H'_j shows that we could take any basis of $\rho^{H_i}, \rho^{H'_j}$ instead of the constructed one. \square

As a consequence we deduce Theorem 2.12: if ρ is rational, and we work up to rational squares, then we do not have to compute $\det(\frac{1}{|H|} \langle , \rangle_1 | \rho^H)$ and $\det(\frac{1}{|H|} \langle , \rangle_2 | \rho^H)$ in the same basis.

2.4. Regulator constants: examples.

Example 2.18. The group $G = S_3$ has 3 irreducible representations, namely $\mathbf{1}$ (trivial), ϵ (sign) and Δ (2-dimensional), and $\mathcal{H} = \{1, C_2, C_3, S_3\}$. The submodule of $\mathbb{Z}\mathcal{H}$ of relations is generated by the following element Θ , with regulator constants

$$\begin{array}{c|ccc} & \mathbf{1} & \epsilon & \Delta \\ \hline \Theta = 2S_3 + 1 - 2C_2 - C_3 & 3 & 3 & 3 \end{array}$$

Hence, if $\text{Gal}(F/K) \cong S_3$ and A/K is principally polarised, then

$$\text{Reg}(A/K)^2 \text{Reg}(A/F) \text{Reg}(A/F^{C_2})^{-2} \text{Reg}(A/F^{C_3})^{-1} = 3^{n_1} 3^{n_\epsilon} 3^{n_\Delta} \cdot \square,$$

with n_ρ the multiplicity of ρ in $A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$. So the parity of $n_1 + n_\epsilon + n_\Delta$ (equivalently of $\text{rk}(A/K) + \text{rk}(A/F^{C_3}) + \text{rk}(A/F^{C_2})$) is “computable”, that is it can be determined from the local invariants using the \square -Conjecture: it is given by

$$\text{ord}_3 C(A/K)^2 C(A/F) C(A/F^{C_2})^{-2} C(A/F^{C_3})^{-1} \pmod{2}.$$

This generalises Example 2.6.

Example 2.19. Take $G = A_5$. Here the irreducible rational representations are $\mathbf{1}, \rho_6, \rho_4, \rho_5$ of dimensions 1, 6, 4 and 5, respectively, and the subgroups are $\mathcal{H} = \{1, C_2, C_3, C_2 \times C_2, C_5, S_3, D_{10}, A_4, A_5\}$. The lattice of relations is generated by 5 elements, and here are the regulator constants:

	$\mathbf{1}$	ρ_6	ρ_4	ρ_5
$\Theta_1 = 1 - 3C_2 + 2C_2 \times C_2$	2	1	1	2
$\Theta_2 = C_2 \times C_2 - 2D_{10} - A_4 + 2A_5$	3	1	3	3
$\Theta_3 = S_3 - D_{10} - A_4 + A_5$	3	1	3	3
$\Theta_4 = 1 - 2C_2 - C_5 + 2D_{10}$	5	5	5	1
$\Theta_5 = C_3 - C_5 - 2A_4 + 2A_5$	15	5	15	3

If E/K is an elliptic curve, it follows that the “computable” combinations are $\mathbf{1} + \rho_5$, $\mathbf{1} + \rho_4 + \rho_5$ and $\mathbf{1} + \rho_6 + \rho_4$. (For a general principally polarised abelian variety only the last two are, see Remark 2.7.) For instance, from $\mathbf{1} + \rho_5$, the parity of $\text{rk}(E/F^{D_{10}})$ can be determined from the local invariants.

It is interesting to note that A_5 is the only group of order < 120 for which there is a computable combination of representations $(\mathbf{1} + \rho_6 + \rho_4)$ where the dimensions add up to an odd number.

Example 2.20. Let $G = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_p)$ for some fixed odd prime p . We write $C_p = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $C_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$. The group G has $p-1$ one-dimensional complex representations whose direct sum is $\text{Ind}^G \mathbf{1}_{C_p}$, and one other $(p-1)$ -dimensional irreducible representation ρ , namely $(\text{Ind}^G \mathbf{1}_{C_{p-1}}) \ominus \mathbf{1}_G$. There is a relation

$$\Theta = 1 - (p-1)C_{p-1} - C_p + (p-1)G.$$

We have $\mathcal{C}(\Theta, \mathbf{1}) = p$ and for \mathbb{Q} -irreducible rational $\sigma \subset (\text{Ind}^G \mathbf{1}_{C_p}) \ominus \mathbf{1}_G$,

$$\sigma^G = \sigma^{C_{p-1}} = 0, \quad \sigma^1 = \sigma^{C_p} = \sigma,$$

so $\mathcal{C}(\Theta, \sigma) = p^{\dim \sigma}$. It remains to determine $\mathcal{C}(\Theta, \rho)$. We have

$$\rho^G = \rho^{C_p} = 0, \quad \rho^1 = \rho, \quad \dim \rho^{C_{p-1}} = 1.$$

If $v \in \rho^{C_{p-1}}$ is non-zero, then $v, gv, \dots, g^{p-2}v$ is a basis for ρ with $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that $v + gv + \dots + g^{p-1}v = 0$ since it is in ρ^{C_p} . Take the Hermitian G -invariant inner product on ρ with $\langle v, v \rangle = 1$, and let us compute the matrix $X = (\langle g^n v, g^m v \rangle)$. By G -invariance we only need $\langle v, g^m v \rangle$, but

$$0 = \langle v, \sum_{m=0}^{p-1} g^m v \rangle = \sum_{m=0}^{p-1} \langle v, g^m v \rangle$$

and the terms in the right-hand side are equal for $m \neq 0$ by C_{p-1} -invariance. So $\langle v, g^m v \rangle = -\frac{1}{p-1}$, and

$$X = \begin{pmatrix} 1 & -\frac{1}{p-1} & \cdots & -\frac{1}{p-1} \\ -\frac{1}{p-1} & 1 & \cdots & -\frac{1}{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{p-1} & -\frac{1}{p-1} & \cdots & 1 \end{pmatrix}.$$

The determinant of X is $\frac{p^{p-2}}{(p-1)^{p-1}}$ and it follows that $\mathcal{C}(\Theta, \rho) = p$. (For $p = 3$ this recovers Example 2.18.)

Corollary 2.21. *Suppose F/K is a Galois extension with Galois group $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{F}_p)$, write M for the fixed field of the commutator subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and L for the fixed field of $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$. For any principally polarised abelian variety A/K with finite $\mathrm{III}(A/F)[p^\infty]$,*

$$\mathrm{rk}(A/K) + \mathrm{rk}(A/M) + \mathrm{rk}(A/L) \equiv \mathrm{ord}_p \frac{C(A/F)}{C(A/M)} \pmod{2}.$$

Proof. Decompose $A(F) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbf{1}^{n_1} \oplus \rho^{n_\rho} \oplus \bigoplus_i \sigma_i^{n_{\sigma_i}}$ into rational irreducibles, with $\sigma_i \subset (\mathrm{Ind}^G \mathbf{1}_{C_p}) \ominus \mathbf{1}_G$. Combining the above example with Corollary 2.13 and Corollary 2.5 (and Remark 2.7), we obtain

$$n_1 + n_\rho + \sum_i n_{\sigma_i} \dim \sigma_i \equiv \mathrm{ord}_p \frac{C(A/F)C(A/K)^{p-1}}{C(A/M)C(A/L)^{p-1}} \pmod{2}.$$

Finally, $\mathrm{rk}(A/K) = n_1$, $\mathrm{rk}(A/M) = n_1 + \sum n_{\sigma_i} \dim \sigma_i$ and $\mathrm{rk}(A/L) = n_1 + n_\rho$. \square

3. TAMAGAWA NUMBERS AND ROOT NUMBERS FOR ELLIPTIC CURVES

3.1. Review of root numbers. We now turn to Tamagawa numbers and their relation to root numbers, in the special case of elliptic curves. We refer to [31, 19] for the classification of root numbers of elliptic curves in odd residue characteristic. Incidentally, while proving Proposition 3.3 we came upon the following formula (case (4)) for local root numbers. It summarises [19] Thm 1.1 (i), (ii) and Remark 1.2 (ii), (iii).

Theorem 3.1. *Let E/K_v be an elliptic curve over a local field. Then*

- (1) $w(E/K_v) = -1$ if $v|\infty$ or E has split multiplicative reduction.
- (2) $w(E/K_v) = 1$ if E has either good or non-split multiplicative reduction.
- (3) $w(E/K_v) = (\frac{-1}{k})$ if E has additive, potentially multiplicative reduction, and the residue field k of K_v has characteristic $p \geq 3$.
- (4) $w(E/K_v) = (-1)^{\lfloor \frac{\text{ord}_v(\Delta)|k|}{12} \rfloor}$, if E has potentially good reduction, and the residue field k of K_v has characteristic $p \geq 5$. Here Δ is the minimal discriminant of E , and $\lfloor x \rfloor$ is the greatest integer $n \leq x$.

Proof. (1,2,3) This follows from the results of [31], [19].

(4) Since $p \geq 5$, we have $\text{ord}_v(\Delta) \in \{0, 2, 3, 4, 6, 8, 9, 10\}$, and it specifies the Kodaira-Néron reduction type of E . Moreover, the class of $|k|$ modulo 24 determines the quadratic residue symbols $(\frac{-1}{k})$, $(\frac{2}{k})$ and $(\frac{3}{k})$. Because in our case $w(E/K_v)$ only depends on the reduction type, $\text{ord}_v(\Delta)$ and these symbols ([19], Thm 1.1, Remark 1.2), this reduces the proof to a (short) finite computation. \square

Remark 3.2. In cases (3) and (4) we have the following results, which are elementary to verify:

- (a) The local root number is unchanged in a totally ramified extension of degree prime to 12, and
- (b) If the residue field has square order, then $w(E/K_v) = 1$.

3.2. The case of $(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix})$ -extensions. As in Example 2.20 and Corollary 2.21, suppose F/K has Galois group $G = (\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}) \subset \text{GL}_2(\mathbb{F}_p)$ for some odd prime p , and let M and L be the fixed fields of $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix})$, respectively. Fix an elliptic curve E/K with an invariant differential ω . For a prime v of K , and $k = K, L, M, F$ set

$$W_v(k) = \prod_{\nu|v} w(E/k_\nu), \quad C_v(k) = \prod_{\nu|v} c_\nu \left| \frac{\omega}{\omega_\nu^o} \right|_\nu,$$

where ω_ν^o is a Néron differential for E at a prime ν of k . For $v|\infty$ we define $W_v(k)$ by the same formula and set $C_v(k) = 1$.

Proposition 3.3. *With fields as above, let E/K be an elliptic curve with a chosen invariant differential ω , and let v be a place of K . If $v|6$ and ramifies in L/K , assume that E is semistable at v . Then*

$$\text{ord}_p \frac{C_v(F) C_v(K)^{p-1}}{C_v(M) C_v(L)^{p-1}} \equiv 0 \pmod{2} \iff W_v(K) W_v(M) W_v(L) = 1.$$

Proof. Clearly the left-hand side is the same as $\text{ord}_p(C_v(F)/C_v(M)) \pmod{2}$. Now consider the following cases depending on the behaviour of v in the (degree p Galois) extension F/M . Note that this extension is ramified if and only if v is ramified in L/K .

Case 1: primes above v in M split in F/M (this includes all Archimedean places). Then $C_v(F) = C_v(M)^p$, so $C_v(F)/C_v(M)$ is a square. Under the

action of the decomposition group D_v at v , the G -sets $G/\text{Gal}(F/L)$ and $(G/\text{Gal}(F/M)) \amalg (G/G)$ are isomorphic. So the number of primes above v with a given ramification and inertial degree is the same in L as in M plus in K . It follows that the local root numbers cancel, $W_v(L) = W_v(K)W_v(M)$.

Case 2: F/M is inert above v . Then v must be totally split in M/K , by the structure of $\text{Gal}(F/K)$. As the number of primes above v in M is even, $C_v(F)$ and $C_v(M)$ are both squares, and $W_v(M) = 1$. Since in this case L_v/K_v is Galois of odd degree, $W_v(L) = W_v(K)$ by Kramer–Tunnell [22], proof of Prop. 3.4.

Case 3: F/M is ramified above v and E is semistable at v . The contributions from ω cancel modulo squares, and $W_v(K) = W_v(L)$. If E has split multiplicative reduction over a prime of M above v , this prime contributes p to $C_v(F)/C_v(M)$ and -1 to the root number. If the reduction is either good or non-split, it contributes to neither.

Case 4: F/M is ramified above $v \nmid 6p$, and E has additive reduction at v . Since $v \nmid p$, there is no contribution from ω , and v is unramified in M/K (again, using the structure of $\text{Gal}(F/K)$ and the fact that totally and tamely ramified Galois extensions of local fields are abelian.) In particular, M has either even number of primes above v or they have even residue field extension. In each case $W_v(M) = 1$ by Remark 3.2(b). It remains to compare $W_v(K), W_v(L)$ and the Tamagawa numbers.

Case 4a: $p \neq 3$. All the Tamagawa numbers are prime to p . Also, because $(p, 12) = 1$ and L_v/K_v is totally ramified, the root numbers $w(E/K_v)$ and $w(E/L_v)$ are equal by Remark 3.2(a).

Case 4b: $p = 3$ and E has reduction type II, II^*, I_0^*, I_n^* (resp. III, III^*) over K_v , and the reduction becomes I_0^*, I_n^* (resp. III, III^*) over L_v . By inspection, the root numbers $w(E/K_v)$ and $w(E/L_v)$ are given by the same residue symbol (this is also clear from [19] 1.1–1.2), so they cancel. Also the Tamagawa numbers are coprime to 3 ([33] IV.9.4).

Case 4c: $p = 3$ and E has reduction IV, IV^* over K_v . The reduction becomes good over L , so $W_v(L) = 1$ and $C_v(F) = 1$. Over K_v the root number is 1 if and only if $-3 \in K_v^{*2}$ ([19] Remark 1.2 (iii)), that is if $\mu_3 \subset K$. This in turn is equivalent to v being split in M/K (K_v has a cubic ramified Galois extension if and only if $\mu_3 \subset K_v$.) Equivalently, there are two primes above v in M and the contribution from the Tamagawa numbers is a square. In the other case, M/K is inert and $C_v(M) = 3$ ([33] IV.9.4, Steps 5, 8).

Case 5: F/M is ramified above $v|p$, $p > 3$ and E has additive reduction at v . By Remark 3.2, $w(E/K_v) = w(E/L_v)$, so we need the parity of $\text{ord}_p(C_v(F)/C_v(M))$ and $W_v(M)$.

Fix a place w over v in M . We can replace ω by the Néron differential of E/M at w , as this changes $C_v(F)/C_v(M)$ by a number of the form λ^p/λ (which is a square), and the parity of its p -adic valuation remains unchanged.

Case 5a: E/M_w has semistable reduction. Our minimal model at w stays minimal in any extension, so there is no contribution from ω . The result follows as in Case 3.

Case 5b: E/M_w has additive reduction. The reduction stays additive over F and, since $p > 3$, all the Tamagawa numbers are prime to p . If M has either even number of primes above v or the residue fields have even degree over \mathbb{F}_p , then $W_v(M) = 1$ (by Remark 3.2) and it also follows that the contributions from ω are squares.

Thus we may assume that M_w/K_v has even ramification degree, in particular E has potentially good reduction at v (for otherwise it would be multiplicative at w). We may also assume that there is an odd number of primes over v in M , and their residue fields are of odd degree over \mathbb{F}_p . By Theorem 3.1 and the fact that $p^2 \equiv 1 \pmod{24}$,

$$w(M_w) = (-1)^{\lfloor \frac{\text{ord}_v(\Delta)|k|}{12} \rfloor} = (-1)^{\lfloor \frac{\text{ord}_v(\Delta)p}{12} \rfloor},$$

and the right-hand side exactly measures the contribution from $|\frac{\omega}{\omega_\nu^\sigma}|_\nu$ for a prime $\nu|w$ of F . The result follows by taking the product over $w|v$. \square

Now we reap the harvest:

Theorem 3.4. *Let p be an odd prime. As above, let F/K have Galois group $G = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_p)$, and let M and L be the fixed fields of $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$, respectively. Let E/K be an elliptic curve such that*

- (1) *The p -primary component $\text{III}(E/F)[p^\infty]$ is finite.*
- (2) *E is semistable at primes $v|6$ that ramify in L/K .*

Then

$$\text{rk}(E/K) + \text{rk}(E/M) + \text{rk}(E/L) \text{ is even} \iff w(E/K)w(E/M)w(E/L) = 1.$$

Proof. As $\text{III}(E/F)[p^\infty]$ is assumed to be finite, by Remark 2.10 the same is true over K , M and L . We now apply the theory from §2 to the relation

$$\text{Ind}_{F/K} \mathbf{1}_F \oplus (\text{Ind}_{K/K} \mathbf{1}_K)^{\oplus p-1} \cong \text{Ind}_{M/K} \mathbf{1}_M \oplus (\text{Ind}_{L/K} \mathbf{1}_L)^{\oplus p-1}.$$

By Corollary 2.21, the sum $\text{rk}(E/K) + \text{rk}(E/M) + \text{rk}(E/L)$ is congruent to $\text{ord}_p C(E/F)/C(E/M)$ modulo 2. By Proposition 3.3, it is even if and only if

$$\prod_{v \text{ place of } K} w(E/K_v) \prod_{v \text{ place of } L} w(E/L_v) \prod_{v \text{ place of } M} w(E/M_v) = 1.$$

\square

3.3. Application to the Parity Conjecture. In [11] Theorem 2 we established the following result:

Theorem 3.5. *Suppose that E/K is semistable at primes above p and has a K -rational isogeny of degree p . If $p = 2$, assume furthermore that E is not supersingular at primes above 2. Then*

$$\text{rk}_p(E/K) \text{ even} \iff w(E/K) = 1.$$

As an application of Theorem 3.4 to $F = K(E[2])$ we can prove a form of the parity conjecture (Conjecture 1.1) without the isogeny assumption:

Theorem 3.6 (=Theorem 1.3). *Let E/K be an elliptic curve. Suppose E is semistable at primes dividing 2 and 3 and not supersingular at primes dividing 2. If $\text{III}(E/K(E[2]))$ has finite 2- and 3-primary parts, then*

$$\text{rk}(E/K) \text{ even} \iff w(E/K) = 1.$$

Proof. Write $F = K(E[2])$, and note that $\text{Gal}(F/K) \subset \text{GL}_2(\mathbb{F}_2) \cong S_3$. By Remark 2.10, the 2- and 3-primary parts of $\text{III}(E/k)$ are finite for $K \subset k \subset F$.

If E has a K -rational 2-torsion point, the result follows from Theorem 3.5. If F/K is cubic, then $\text{rk}(E/K)$ and $\text{rk}(E/F)$ have the same parity, and also $w(E/K) = w(E/F)$, so the result again follows.

We are left with the case when $\text{Gal}(F/K) \cong S_3 \cong \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_3)$. Let M be the quadratic extension of K in F and L one of the cubic ones. By the above argument, we know that

$$\begin{aligned} \text{rk}(E/M) \text{ even} &\iff w(E/M) = 1, \\ \text{rk}(E/L) \text{ even} &\iff w(E/L) = 1. \end{aligned}$$

On the other hand, by Theorem 3.4 with $p = 3$,

$$\text{rk}(E/K) + \text{rk}(E/M) + \text{rk}(E/L) \text{ is even} \iff w(E/K)w(E/M)w(E/L) = 1.$$

□

Remark 3.7. Instead of assuming that III is finite in the theorem one may give a statement about Selmer ranks, by replacing the use of Theorem 3.4 by Corollary 4.12 in the proof. When $\text{Gal}(F/K) \cong S_3$, the parity of

$$\text{rk}_3(E/K) + (\text{rk}_3(E/M) - \text{rk}_2(E/M)) + (\text{rk}_3(E/L) - \text{rk}_2(E/L))$$

is given by the root number $w(E/K)$, unconditionally on finiteness of III . In all other cases it is the parity of $\text{rk}_2(E/K)$.

We would also like to remark that if Theorem 3.5 can be extended to curves with arbitrary reduction at $v|2$, and Proposition 3.3 to extensions where additive primes $v|6$ are allowed to ramify, the parity conjecture for all elliptic curves over number fields would follow from finiteness of III .

4. SELMER GROUPS

Hitherto our main tool was Corollary 2.5, relating regulators to Tamagawa numbers assuming that III is finite. In §4.1 we extend this to an unconditional statement about Selmer ranks. We get our results (Theorem 4.3 and Corollary 4.5) by tweaking Tate–Milne’s proof of the isogeny invariance of the BSD quotient ([25], §1.7). The quotient of regulators becomes replaced by a quantity Q measuring the effect of an isogeny on Selmer groups. In §4.2 we review how to construct isogenies between products of Weil restrictions of an abelian variety, and in §4.3 address the question of turning Q into Selmer ranks (somewhat analogous to turning regulators into Mordell–Weil ranks.) This can be done in fair generality (Theorem 4.7, Corollary 4.8), and we illustrate it for S_n -extensions (Ex. 4.9), $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ -extensions (§4.4), and dihedral extensions (§4.5). As a final application, in §4.6 we establish the p -parity conjecture for elliptic curves over \mathbb{Q} .

4.1. Invariance of the BSD-quotient for Selmer groups.

Definition 4.1. For an isogeny $\psi : A \rightarrow B$ of abelian varieties over K , let

$$\begin{aligned} Q(\psi) = & |\text{coker}(\psi : A(K)/A(K)_{\text{tors}} \rightarrow B(K)/B(K)_{\text{tors}})| \times \\ & \times |\ker(\psi : \text{III}(A)_{\text{div}} \rightarrow \text{III}(B)_{\text{div}})|, \end{aligned}$$

where III_{div} denotes the divisible part of III .

Lemma 4.2. *$Q(\psi)$ is finite and satisfies the following properties:*

- (1) $Q(\psi' \psi) = Q(\psi)Q(\psi')$ if $\psi : A \rightarrow B$ and $\psi' : B \rightarrow C$ are isogenies.
- (2) $Q(\psi \oplus \psi') = Q(\psi)Q(\psi')$ if $\psi : A \rightarrow B$ and $\psi' : A' \rightarrow B'$ are isogenies.
- (3) $Q(\psi) = p^{\text{rk}_p(A/K)}$ if $\psi : A \rightarrow A$ is multiplication by p .
- (4) If $\deg \psi$ is prime to p , then so is $Q(\psi)$.

Proof. (2), (3) clear. (1) follows from the fact that $\psi : \text{III}(A)_{\text{div}} \rightarrow \text{III}(B)_{\text{div}}$ is surjective, and $\psi : A(K)/A(K)_{\text{tors}} \rightarrow B(K)/B(K)_{\text{tors}}$ is injective. Next, consider the conjugate isogeny $\psi^c : B \rightarrow A$, so that $\psi^c \psi$ is the multiplication by $\deg \psi$ map on A . From (1) and (3), $Q(\psi^c)Q(\psi)$ is finite, so $Q(\psi)$ is finite. (4) also follows. \square

Theorem 4.3. Let $X, Y/K$ be abelian varieties given with exterior forms ω_X, ω_Y . Suppose $\phi : X \rightarrow Y$ is an isogeny and $\phi^t : Y^t \rightarrow X^t$ its dual. Writing $\text{III}_0(X/K)$ for $\text{III}(X/K)$ modulo its divisible part and

$$\Omega_X = \prod_{\substack{v|\infty \\ \text{real}}} \int_{X(K_v)} |\omega_X| \cdot \prod_{\substack{v|\infty \\ \text{complex}}} \int_{X(K_v)} \omega_X \wedge \bar{\omega}_X$$

and similarly for Y , we have

$$\frac{|Y(K)_{\text{tors}}|}{|X(K)_{\text{tors}}|} \frac{|Y^t(K)_{\text{tors}}|}{|X^t(K)_{\text{tors}}|} \frac{C(X/K)}{C(Y/K)} \frac{\Omega_X}{\Omega_Y} \prod_{p|\deg \phi} \frac{|\text{III}_0(X)[p^\infty]|}{|\text{III}_0(Y)[p^\infty]|} = \frac{Q(\phi^t)}{Q(\phi)}.$$

Proof. Recall that ω_X and ω_Y enter into the definition of $C(X/K)$ and $C(Y/K)$. The left-hand side of the asserted equation is independent of the choices of ω_X and ω_Y by the product formula, so choose $\omega_X = \phi^* \omega_Y$. Note also that ϕ is an isomorphism between the p -primary parts of $\text{III}_0(X)$ and $\text{III}_0(Y)$ for $p \nmid \deg \phi$, so the product involving III may be taken over any sufficiently large set of primes. (In fact, it is simply $|\text{III}(X)|/|\text{III}(Y)|$ if both groups are finite.)

Now we follow closely Tate–Milne’s proof in [25], §1.7. If f is a homomorphism of abelian groups with finite kernel and cokernel, write

$$z(f) = \frac{|\ker f|}{|\text{coker } f|}.$$

For $k \supset K$ denote by $\phi(k) : X(k) \rightarrow Y(k)$ the map induced by ϕ on k -rational points, and similarly for ϕ^t . For a sufficiently large set of places S of K ([25] I.(7.3.1)),

$$\prod_{v \in S} z(\phi(K_v)) = \frac{z(\phi(K))}{z(\phi^t(K))} \frac{|\text{III}[\phi^t]|}{|\text{III}[\phi]|}.$$

Moreover, $z(\phi(K_v))$ is the contribution from v to $C(Y/K)/C(X/K)$ for finite places, and the quotient of the corresponding integrals for infinite places with our choice for ω_X, ω_Y . (Milne also relates $z(\phi(K))/z(\phi^t(K))$ to the torsion and the regulators and, assuming finiteness of the Tate-Shafarevich groups, $|\text{III}[\phi^t]|/|\text{III}[\phi]|$ to $|\text{III}(Y)|/|\text{III}(X)|$. This gives the usual formula for the isogeny invariance of the BSD-quotient.)

It remains to show that for every prime p ,

$$\text{ord}_p \frac{z(\phi^t(K))}{z(\phi(K))} \frac{|\text{III}[\phi]|}{|\text{III}[\phi^t]|} = \text{ord}_p \frac{Q(\phi)}{Q(\phi^t)} \frac{|Y^t(K)_{\text{tors}}|}{|X^t(K)_{\text{tors}}|} \frac{|Y(K)_{\text{tors}}|}{|X(K)_{\text{tors}}|} \frac{|\text{III}_0(X)[p^\infty]|}{|\text{III}_0(Y)[p^\infty]|}.$$

Take an integer $N = p^m$ large enough, so that it kills both the p -power torsion in $X(K)$ and $Y(K)$ and the p -parts of $\text{III}_0(X)$ and $\text{III}_0(Y)$. Applying Lemma 4.2 (1, 3),

$$Q(p^m\phi) = p^{m \text{rk}_p(X/K)} Q(\phi), \quad Q(p^m\phi^t) = p^{m \text{rk}_p(Y^t/K)} Q(\phi^t).$$

Since X, Y and their duals all have the same p^∞ -Selmer rank (they are all isogenous), it suffices to verify the claim for $\psi = p^m\phi$. But

$$\begin{aligned} \text{ord}_p |\text{III}[\psi]| &= \text{ord}_p |\text{III}_0(X)| \cdot |\ker(\psi|_{\text{III}(X)_{\text{div}}})| \\ \text{ord}_p |\ker(\psi(K))| &= \text{ord}_p |X(K)_{\text{tors}}| \\ \text{ord}_p |\text{coker}(\psi(K))| &= \text{ord}_p |Y(K)_{\text{tors}}| \cdot |\text{coker}(\frac{X(K)}{X(K)_{\text{tors}}} \xrightarrow{\psi} \frac{Y(K)}{Y(K)_{\text{tors}}})|, \end{aligned}$$

and similarly for ψ^t . Combining these together yields the assertion. \square

Remark 4.4. For $\phi : E \rightarrow E'$ a cyclic isogeny of degree p , this gives

$$\frac{C(E/K)}{C(E'/K)} \frac{\Omega_E}{\Omega_{E'}} \equiv \frac{Q(\phi^t)}{Q(\phi)} \equiv Q(\phi^t)Q(\phi) = Q([p]) = p^{\text{rk}_p(E/K)} \pmod{\mathbb{Q}^{*2}},$$

which is a formula of Cassels (see Birch [2] or Fisher [14]).

Corollary 4.5. Let E/K be an elliptic curve with a chosen K -differential ω . Suppose $L_i/K, L'_j/K$ are finite extensions such that

$$X = \prod_i W_{L_i/K}(E), \quad Y = \prod_j W_{L'_j/K}(E)$$

are isogenous. If $\phi : X \rightarrow Y$ is an isogeny and ϕ^t its dual, then

$$\frac{\prod_i C(E/L_i)}{\prod_j C(E/L'_j)} \equiv Q(\phi^t)Q(\phi) \pmod{\mathbb{Q}^{*2}}.$$

The same is true if E is replaced by a principally polarised abelian variety over K , possibly up to a factor of 2 if the polarisation is not induced by a K -rational divisor.

Proof. Inducing exterior forms on $W_{L_i/K}(E)$ and $W_{L'_j/K}(E)$ by ω , we have $\Omega_X = \Omega_Y$. Moreover, $X \cong X^t$, $Y \cong Y^t$ and the p -primary parts of $\text{III}/\text{III}_{\text{div}}$ have square order by Cassels–Tate pairing. \square

4.2. Isogenies between products of Weil restrictions. To make Corollary 4.5 explicit, recall from Milne’s [24] §2 how to construct isogenies

$$X = \prod_i W_{L_i/K}(A) \xrightarrow{\phi} \prod_j W_{L'_j/K}(A) = Y$$

for a principally polarised abelian variety A/K . For an extension L/K write $G_L = \text{Gal}(\bar{K}/L)$. Suppose $\oplus_i \text{Ind}_{L_i/K} \mathbf{1}_{L_i} \cong \oplus_j \text{Ind}_{L'_j/K} \mathbf{1}_{L'_j}$, and consider

$$M_X = \oplus_i \mathbb{Z}[G_K/G_{L_i}], \quad M_Y = \oplus_j \mathbb{Z}[G_K/G_{L'_j}].$$

These are G_K -modules, and satisfy $M_X \otimes \mathbb{Q} \cong M_Y \otimes \mathbb{Q}$. In general, if M is such a module with a given identification $M \cong \mathbb{Z}^n$ (as an abelian group), the composition

$$s : G_K \longrightarrow \text{Aut}_{\mathbb{Z}}(M) = \text{Aut}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Z}) \longrightarrow \text{Aut}(A^n)$$

is an element of $H^1(G_K, \text{Aut}_{\bar{K}}(A^n))$. It corresponds to a unique form of A^n over K , that is an abelian variety over K such that A^n is isomorphic to it via an isomorphism ψ defined over \bar{K} . (The relation between ψ and s is $s(\sigma) = \psi^{-1}\psi^\sigma$.) Milne denotes this form $A \otimes M$, and with this notation $X = A \otimes M_X$ and $Y = A \otimes M_Y$.

Next, a principal polarisation $\lambda : A \rightarrow A^t$ induces one on A^n . So we can view $(A \otimes M)^t$ as a form of A^n , which is seen to be the same as $A \otimes \text{Hom}(M, \mathbb{Z})$. If M is a permutation module, there is a natural isomorphism $M \cong \text{Hom}(M, \mathbb{Z})$, and it induces a principal polarisation on $A \otimes M$.

Now suppose $f : M_X \rightarrow M_Y$ is an isogeny of G_K -modules (a G_K -invariant injection with finite cokernel), viewed as an $n \times n$ -matrix with integer coefficients. Then

$$\phi_f : X \xrightarrow{\psi_X^{-1}} A^n \xrightarrow{f} A^n \xrightarrow{\psi_Y} Y$$

is an isogeny of degree $|\det f|^{2 \dim A}$ defined over K ([24], Prop. 6a), with the dual isogeny

$$\phi_f^t : X^t \xleftarrow{(\psi_X^{-1})^t} (A^n)^t \xleftarrow{f^t} (A^n)^t \xleftarrow{\psi_Y^t} Y^t.$$

With respect to the above polarisations, f^t is the transpose of f (see e.g. [10] §1.6, esp. Lemma 3).

To summarise: the natural identifications $M_X \cong \mathbb{Z}^n$ and $M_Y \cong \mathbb{Z}^n$ induce principal polarisations on X and Y ; an isogeny $f : M_X \rightarrow M_Y$ induces an isogeny $\phi_f : X \rightarrow Y$ of degree $|\det f|^{2 \dim A}$; suppressing the principal polarisations, $(\phi_f)^t \phi_f = \phi_{f^t f}$ where f^t is the transposed matrix and $(\phi_f)^t$ is the dual isogeny. Thus, the right-hand side in Corollary 4.5 for ϕ_f becomes Q of an explicit endomorphism $\phi_{f^t f}$ of X .

4.3. Determining Q . Fix a principally polarised abelian variety A/K and a finite extension F/K with Galois group G .

Let $\{\rho_k\}_k$ be the set of \mathbb{Q} -irreducible rational representations of G . For $\rho \in \{\rho_k\}$ we will write $\text{rk}_p(A, \rho)$ for the p^∞ -Selmer rank of $A \otimes \Lambda$, where $\mathbb{Z}^{\dim \rho} \cong \Lambda \subset \rho$ is any G -invariant lattice. Since the Selmer rank is the

same for isogenous abelian varieties, this is independent of the choice of the lattice. Moreover, for $K \subset L \subset F$ with $\mathbb{Q}[G/\text{Gal}(F/L)] \cong \bigoplus \rho_k^{n_k}$,

$$\text{rk}_p(A/L) = \sum_k n_k \text{rk}_p(A, \rho_k).$$

We want to express Q in terms of these Selmer ranks.

Lemma 4.6. *Let V be a rational representation of G , and $f \in \text{Aut}_G(V)$. For any G -invariant lattice Λ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = V$ there is an integer $m \geq 1$ such that mf preserves Λ , and*

$$Q(f) := Q(\phi_{mf:\Lambda \rightarrow \Lambda})/Q(\phi_{m:\Lambda \rightarrow \Lambda}) \in \mathbb{Q}^*$$

is independent of Λ and m . It satisfies the following properties:

- (1) $Q(f'f) = Q(f')Q(f)$ for $f, f' \in \text{Aut}_G(V)$.
- (2) $Q(f \oplus f') = Q(f)Q(f')$ for $f \in \text{Aut}_G(V), f' \in \text{Aut}_G(V')$.
- (3) $Q(f) = \prod_k p^{n_k \text{rk}_p(A, \rho_k)}$ if $f : V \rightarrow V$ is multiplication by p , with $V = \bigoplus_k \rho_k^{\oplus n_k}$ the decomposition into rational irreducibles.
- (4) Suppose $f \in \text{Aut}_G(V)$ has an irreducible minimal polynomial with p -adically integral coefficients and $p \nmid \det f$. Then $\text{ord}_p Q(f) = 0$.

Proof. Independence of m follows from the (obvious) special case $m|m'$. Similarly we can reduce to the case $\Lambda_1 \subset \Lambda_2$ with $m_1 = m_2 = m$. Let $\iota : \Lambda_1 \rightarrow \Lambda_2$ be the inclusion map, and $n \geq 1$ an integer such that $n\Lambda_2 \subset \Lambda_1$. Then

$$n \circ (\phi_{mf:\Lambda_2 \rightarrow \Lambda_2}) = \iota \circ (\phi_{mf:\Lambda_1 \rightarrow \Lambda_1}) \circ (n\iota^{-1}).$$

The independence of Λ now follows from Lemma 4.2 (1).

(1-3) are immediate from Lemma 4.2.

(4) Let $m(x)$ be the minimal polynomial of f . After scaling f by an integer coprime to p if necessary, we may assume that $m(x)$ has integer coefficients. Let $\alpha \in \bar{\mathbb{Q}}$ be a root of m , and let $\mathcal{K} = \mathbb{Q}(\alpha)$. Note that α is an algebraic integer, $\alpha \in \mathcal{O}_{\mathcal{K}}$.

Via the action of f and of G , the representation ρ is naturally a $\mathcal{K}[G]$ -module. Take a G -invariant full $\mathcal{O}_{\mathcal{K}}$ -lattice Λ . (It exists, since one may take any full $\mathcal{O}_{\mathcal{K}}$ -lattice, and generate a lattice by its G -conjugates.) In particular, it is a full G -invariant \mathbb{Z} -lattice preserved by f , so $Q(f) = Q(\phi_f)/Q(\phi_{\text{id}})$ is an integer. By Lemma 4.2(4) it is prime to p , as p does not divide $\deg \phi_f = |\det f|^{2 \dim A}$. \square

Theorem 4.7. *Let $V \cong \rho^{\oplus n}$, with ρ a \mathbb{Q} -irreducible rational representation of G , and let $f \in \text{Aut}_G(V)$. Suppose p is a prime such that either*

- (1) ρ is irreducible as a $\mathbb{Q}_p[G]$ -representation, or
- (2) for every irreducible factor $m(x) \mid \det(f - xI) \in \mathbb{Q}[x]$, all of the roots of $m(x)$ in $\bar{\mathbb{Q}}_p$ have the same valuation.

Then

$$\text{ord}_p Q(f) \equiv \frac{\text{ord}_p \det f}{\dim \rho} \text{rk}_p(A, \rho) \pmod{2}.$$

Proof. (2) First, we can break up V as follows. Let

$$\det(f - xI) = \prod_k m_k(x)^{n_k}$$

be the factorisation into irreducibles. Then $V_k = \ker m_k(f)^{n_k}$ is G -invariant because f commutes with the action of G (so G preserves its generalised eigenspaces). Since $V = \bigoplus V_k$, by Lemma 4.6 (2) it suffices to prove the statement with $V = V_k$ and $\det(f - xI)$ a power of an irreducible polynomial $m(x)$.

Suppose all the roots of $m(x)$ in $\bar{\mathbb{Q}}_p$ have the same valuation. Write

$$f^{\dim V} = p^{\text{ord}_p \det f} \cdot f',$$

so that the roots of the characteristic polynomial of f' in $\bar{\mathbb{Q}}_p$ are all p -adic units. Then $\text{ord}_p Q(f') = 0$ by Lemma 4.6 (4), and the claim follows by Lemma 4.6 (1, 3).

(1) Fix an identification $V = \rho^{\oplus n}$. As $D = \text{End}_G(\rho)$ is a skew field, we can put f into a block-diagonal form by multiplying it on the left and on the right by $n \times n$ matrices with values in D that are (a) permutation matrices and (b) identity plus some element of D in (i, j) -th place ($i \neq j$). (This is just the usual Gaussian elimination over a skew field.) All of these elementary matrices have $Q = 1$ (they are either of finite order or commutators) and $\det = \pm 1$, so we are reduced to the case $V = \rho$.

We claim that for V irreducible over \mathbb{Q}_p , the eigenvalues of f in $\bar{\mathbb{Q}}_p$ have the same valuation (so (2) applies). But otherwise the minimal polynomial of f is reducible over \mathbb{Q}_p , and we can decompose V over \mathbb{Q}_p as above, contradicting the irreducibility. \square

Corollary 4.8. *Let $K \subset L_i, L'_j \subset F$ be finite extensions with F/K Galois with Galois group G . Suppose there is an isogeny of $\mathbb{Z}[G]$ -modules*

$$f : \prod_i \mathbb{Z}[G/H_i] \longrightarrow \prod_j \mathbb{Z}[G/H'_j],$$

where $H_i = \text{Gal}(F/L_i)$ and $H'_j = \text{Gal}(F/L'_j)$. Assume furthermore that on every isotypical component ρ^{n_ρ} of $\prod_i \mathbb{Q}[G/H_i]$ the automorphism $f^t f$ satisfies either (1) or (2) of Theorem 4.7. Then for every elliptic curve E/K with a chosen K -differential ω ,

$$\text{ord}_p \frac{\prod_i C(E/L_i)}{\prod_j C(E/L'_j)} \equiv \sum_{\rho} \frac{\text{ord}_p \det(f^t f \mid \rho^{n_\rho})}{\dim \rho} \text{rk}_p(E, \rho) \pmod{2},$$

the sum taken over the distinct \mathbb{Q} -irreducible rational representations of G .

Remark. This also holds for principally polarised abelian varieties for odd p , and for $p = 2$ provided the polarisation is induced by a K -rational divisor.

Here are some special cases when the theorem applies:

Example 4.9. If $G = S_n$, then every \mathbb{Q} -irreducible rational representation is absolutely irreducible, so the condition (1) of Theorem 4.7 always holds. Thus the corollary applies for every isogeny and all p .

Example 4.10. For all groups with $|G| \leq 55$, every relation of permutation representations and every prime p , we have checked that there is always an isogeny satisfying one of the conditions of Theorem 4.7 on every isotypic component. In each case, the coefficient of $\text{rk}_p(A, \rho)$ agrees with the regulator constant of §2.3. Is this true in general?²

4.4. Example: Selmer ranks for $(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix})$ -extensions. As an illustration, we extend Corollary 2.21 to Selmer ranks:

Theorem 4.11. Let p be an odd prime. Suppose F/K has Galois group $G = (\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}) \subset \text{GL}_2(\mathbb{F}_p)$, and let M and L be the fixed fields of $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix})$, respectively. For every principally polarised abelian variety A/K ,

$$\text{rk}_p(A/K) + \text{rk}_p(A/L) + \text{rk}_p(A/M) \equiv \text{ord}_p \frac{C(A/F)}{C(A/M)} \pmod{2}.$$

Proof. Consider the abelian varieties

$$X = W_{L/K}(A)^{p-1} \times W_{M/K}(A), \quad Y = A^{p-1} \times W_{F/K}(A).$$

By Corollary 4.5, it suffices to show that

$$\text{rk}_p(A/K) + \text{rk}_p(A/L) + \text{rk}_p(A/M) \equiv \text{ord}_p Q(\phi^t \phi) \pmod{2}$$

for some isogeny $\phi : X \rightarrow Y$. Write $\text{Gal}(F/M) = \langle g \rangle$, $\text{Gal}(F/L) = \langle h \rangle$ with $g^p = 1 = h^{p-1}$, and introduce permutation modules

$$\begin{aligned} \mathbb{Z}_K &= \mathbb{Z}[G/G] &= \mathbb{Z} \\ \mathbb{Z}_L &= \mathbb{Z}[G/\langle h \rangle] &= \oplus \mathbb{Z}g^i & 0 \leq i \leq p-1 \\ \mathbb{Z}_M &= \mathbb{Z}[G/\langle g \rangle] &= \oplus \mathbb{Z}h^j & 0 \leq j \leq p-2 \\ \mathbb{Z}_F &= \mathbb{Z}[G] &= \oplus \mathbb{Z}g^ih^j. \end{aligned}$$

Consider

$$\begin{aligned} V_1 &= \mathbb{Z}_Lx_1 \oplus \dots \oplus \mathbb{Z}_Lx_{p-1} \oplus \mathbb{Z}_Mx_p, \\ V_2 &= \mathbb{Z}_Ky_1 \oplus \dots \oplus \mathbb{Z}_Ky_{p-1} \oplus \mathbb{Z}_Fy_p, \end{aligned}$$

and take the G -invariant map $f : V_1 \rightarrow V_2$ determined by

$$\begin{aligned} x_1 &\mapsto y_1 &+ \sum_j h^j y_p \\ x_k &\mapsto y_1 - y_k &+ \sum_j h^j (1 - g^{1-k}) y_p & (k = 2, \dots, p-1) \\ x_p &\mapsto y_1 + \dots + y_{p-1} &- \sum_i h^{-1}g^i y_p. \end{aligned}$$

It is easy to check that it is well-defined, and moreover, written as a matrix on the chosen \mathbb{Z} -basis of V_1 and V_2 ,

$$|\det f| = (p^2 - p + 1)p^{\frac{p(p-1)}{2}-1},$$

in particular non-zero. So f induces an isogeny $\phi_f : X \rightarrow Y$ (§4.2). Next, ϕ_f^t is given by the transposed matrix (§4.2 again), and the composition $f^t f$ by

$$\begin{aligned} x_1 &\mapsto \sum_{i \neq 0} (g^i x_1 + px_i) \\ x_k &\mapsto px_k + \sum_{i \neq 0} px_i & (k = 2, \dots, p-1) \\ x_p &\mapsto (p + \sum_j (p-1)h^j) x_p. \end{aligned}$$

²Yes, see [12] which provides an analogue of regulator constants for Selmer groups.

(As an example, for $p = 3$ the maps f and $f^t f$ are

$$f : \left(\begin{array}{ccc|ccc|cc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \end{array} \right) \quad f^t f : \left(\begin{array}{ccc|ccc|cc} 3 & 1 & 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 3 & 0 & 0 \\ \hline 3 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 \end{array} \right)$$

with respect to the bases $\{x_1, gx_1, g^2x_1, x_2, gx_2, g^2x_2, x_3, hx_3\}$ of V_1 and $\{y_1, y_2, y_3, gy_3, g^2y_3, hy_3, hg_3, hg^2y_3\}$ of V_2 .)

Clearly $f^t f = \alpha_1 \oplus \alpha_2$, with α_1 an endomorphism of \mathbb{Z}_L^{p-1} and α_2 of \mathbb{Z}_M . To prove the theorem it suffices to show that

$$\begin{aligned} \text{ord}_p Q(\alpha_1) &= (p-2) \text{rk}_p(W_{L/K}(A)/K), \\ \text{ord}_p Q(\alpha_2) &= \text{rk}_p(W_{M/K}(A)/K) - \text{rk}_p(A/K). \end{aligned}$$

The map α_1 is the composition of $\text{id} \oplus p \oplus \cdots \oplus p$ with an endomorphism of determinant $p^2 - p + 1$. Each multiplication by p on a copy of $W_{L/K}(A)$ contributes $p^{\text{rk}_p(W_{L/K}(A)/K)}$ to $Q(\alpha_1)$, and the remaining endomorphism contributes nothing (Lemma 4.2).

As for α_2 , consider

$$\alpha_2 \oplus [p] : \mathbb{Z}_M z_1 \oplus \mathbb{Z}_K z_2 \longrightarrow \mathbb{Z}_M z_1 \oplus \mathbb{Z}_K z_2.$$

It is easy to check that

$$\alpha_3 \circ (\alpha_2 \oplus [p]) = ([p] \oplus \text{id}) \circ \alpha_4,$$

with

$$\alpha_3 : \left\{ \begin{array}{l} z_1 \rightarrow z_1 + \sum_j h^j z_1 \\ z_2 \rightarrow (p-1) \sum_j h^j z_2 \end{array} \right. , \quad \alpha_4 : \left\{ \begin{array}{l} z_1 \rightarrow z_1 + p \sum_j h^j z_1 + z_2 \\ z_2 \rightarrow (p-1)(p^2 - p + 1) \sum_j h^j z_1 \end{array} \right. .$$

Furthermore, $\det \alpha_3$ and $\det \alpha_4$ are prime to p , and it follows that $\text{ord}_p Q(\alpha_2)$ is $\text{rk}_p(W_{M/K}(A)/K) - \text{rk}_p(A/K)$, as asserted. \square

Using the result on the local Tamagawa numbers in this extension (Proposition 3.3) we obtain the following strengthening of Theorem 3.4.

Corollary 4.12. *Let p be an odd prime. As above, let F/K have Galois group $G = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_p)$, and let M and L be the fixed fields of $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$, respectively. For every elliptic curve E/K with semistable reduction at the primes $v \mid 6$ that ramify in L/K ,*

$$\text{rk}_p(E/K) + \text{rk}_p(E/M) + \text{rk}_p(E/L) \text{ is even} \Leftrightarrow w(E/K)w(E/M)w(E/L) = 1.$$

This can be used to study the ranks of elliptic curves in an infinite “false Tate curve extension” with Galois group $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \text{GL}_2(\mathbb{Z}_p)$. Arithmetic of elliptic curves (ordinary at p) in such extensions has been studied in the context of non-commutative Iwasawa theory, see e.g. [17, 7, 8].

Thus, fix a number field K , an odd prime p , and $\alpha \in K^*$. We are interested in the extensions $K(\sqrt[p^n]{\alpha})$ and $K(\mu_{p^n}, \sqrt[p^n]{\alpha})$ of K . We will assume that their degree is maximal possible, i.e. p^n and $(p-1)p^{2n-1}$, respectively.

Proposition 4.13. *Let E/K be an elliptic curve for which the parity of the p^∞ -Selmer rank agrees with the root number over K and over $K(\mu_p)$, and semistable at all primes $v|6$ that ramify in $K(\sqrt[p^n]{\alpha})/K$. Then*

$$\text{rk}_p(E/K(\sqrt[p^n]{\alpha})) \text{ is even} \iff w(E/K(\sqrt[p^n]{\alpha})) = 1.$$

Proof. For brevity, let us write $L_i = K(\sqrt[p^i]{\alpha})$ and $F_i = K(\mu_p, \sqrt[p^i]{\alpha})$ for $i \geq 0$. We prove the result by induction on i , by showing that if the parity of the p^∞ -Selmer rank agrees with the root number over L_{i-1} and F_{i-1} then it does so over L_i and F_i (for $1 \leq i \leq n$).

The extension F_i/F_{i-1} is Galois of odd degree, so $w(E/F_i) = w(E/F_{i-1})$. By Corollary 4.15 below, the parity of the p^∞ -Selmer rank is also unchanged in this extension. The fact that the parity of the p^∞ -Selmer rank agrees with the root number over L_i follows from Corollary 4.12 applied to the extension F_i/L_{i-1} . \square

The following is a standard result on the behaviour of Selmer groups in Galois extensions. We give a brief proof for lack of a reference. Write Sel_{p^n} and Sel_{p^∞} for the p^n - and p^∞ -Selmer groups, and set

$$\mathcal{X}_p(E/K) = \text{Hom}(\text{Sel}_{p^\infty}(E/K), \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The p^∞ -Selmer rank of E/K is the same as the dimension of $\mathcal{X}_p(E/K)$ as a \mathbb{Q}_p -vector space.

Lemma 4.14. *Let E/K be an elliptic curve, and let F/K be a finite Galois extension with Galois group G . Then*

$$\text{rk}_p(E/K) = \dim_{\mathbb{Q}_p} \mathcal{X}_p(E/F)^G.$$

Proof. The restriction map from $H^1(K, E[p^n])$ to $H^1(F, E[p^n])^G$ induces a map $\text{Sel}_{p^n}(E/K) \rightarrow \text{Sel}_{p^n}(E/F)^G$ whose kernel and cokernel are killed by $|G|^2$. Taking direct limits gives a map from $\text{Sel}_{p^\infty}(E/K)$ to $\text{Sel}_{p^\infty}(E/F)^G$, whose kernel and cokernel are killed by $|G|^2$. The result follows by taking duals and tensoring with \mathbb{Q}_p . \square

A cyclic group of order p has only two \mathbb{Q}_p -irreducible p -adic representations, the trivial one and one of dimension $p-1$. Thus,

Corollary 4.15. *The parity of the p^∞ -Selmer rank is unchanged in cyclic p -extensions.*

Example 4.16. Let $p = 3$ and consider the elliptic curve

$$E : y^2 + xy = x^3 - x^2 - 2x - 1 \quad (49A1).$$

It has additive reduction of Kodaira type III at 7 and is supersingular at 3.

For a false Tate curve extension, we take $\mathbb{Q}(\mu_{3^n}, \sqrt[3^n]{m})$ for some cube free $m > 1$. Using 3-descent for E and its quadratic twist by -3 over \mathbb{Q} , it is easy

to see that $\text{rk}(E/\mathbb{Q}) = \text{rk}_3(E/\mathbb{Q}) = 0$ and $\text{rk}(E/\mathbb{Q}(\mu_3)) = \text{rk}_3(E/\mathbb{Q}(\mu_3)) = 1$, both in agreement with the root numbers.

By Proposition 4.13, the 3^∞ -Selmer rank of E over $L_n = K(\sqrt[p^n]{m})$ agrees with the root number, which equals $(-1)^n$ for every m (-1 from $v|7$ and $(-1)^{n-1}$ from $v|\infty$). Because the Selmer rank is non-decreasing in extensions (e.g. by Lemma 4.14), the 3^∞ -Selmer rank must be at least n over L_n . In fact, using Lemma 4.14 and that $\text{Ind}_{L_n/\mathbb{Q}} \mathbf{1}_{L_n} \ominus \text{Ind}_{L_{n-1}/\mathbb{Q}} \mathbf{1}_{L_{n-1}}$ is irreducible, it is easy to see that the 3^∞ -Selmer rank over $\mathbb{Q}(\mu_{3^n}, \sqrt[3^n]{m})$ is at least 3^n .

4.5. Example: Dihedral groups. As another illustration, we consider dihedral groups to obtain similar results to [23], e.g. Theorem 8.5. For simplicity, we will only look at D_{2p} with p an odd prime.

Proposition 4.17. *Suppose $\text{Gal}(F/K) = D_{2p}$ with p an odd prime, and pick extensions M/K and L/K in F of degree 2 and p , respectively. For every principally polarised abelian variety A/K ,*

$$\text{rk}_p(A/M) + \frac{2}{p-1}(\text{rk}_p(A/L) - \text{rk}_p(A/K)) \equiv \text{ord}_p \frac{C(A/F)}{C(A/M)} \pmod{2}.$$

Proof. First let $G = D_{2n} = \langle g, h \mid g^n = h^2 = hg = 1 \rangle$ for a general n , and write $n = 2m + \delta$ with $\delta \in \{0, 1\}$. Take the permutation modules

$$\begin{aligned} V_1 &= v_1 \mathbb{Z}[G/\langle g^{-1}h \rangle] \oplus v_2 \mathbb{Z}[G/\langle g^{-2}h \rangle] \oplus v_3 \mathbb{Z}[G/\langle g \rangle], \\ V_2 &= w_1 \mathbb{Z}[G/G] \oplus w_2 \mathbb{Z}[G/G] \oplus w_3 \mathbb{Z}[G]. \end{aligned}$$

Consider $f : V_1 \rightarrow V_2$ and $f^t f : V_1 \rightarrow V_1$ given respectively by

$$\begin{array}{ll} v_1 \mapsto (1 + g^{-1}h)w_3 & v_1 \mapsto 2v_1 \\ v_2 \mapsto w_2 + g^{m-2}(1 - g^{1+\delta})(g - h)w_3 & v_2 \mapsto (4 - 2g^{1+\delta} - 2g^{-1-\delta} + \sum_{i=0}^{n-1} g^i)v_2 \\ v_3 \mapsto w_1 + \sum_{i=0}^{n-1} g^i(1 - h)w_3 & v_3 \mapsto ((2n+1) - (2n-1)h)v_3 \end{array}$$

Note that $f^t f$ decomposes naturally as $\alpha_1 \oplus \alpha_2 \oplus \alpha_3$ on V_1 , and that α_2 is given on the basis $\{v_2, gv_2, \dots, g^{n-1}v_2\}$ by the matrix

$$\begin{array}{c} \left(\begin{array}{ccccccccc} 5 & -1 & 1 & 1 & 1 & \dots & 1 & 1 & -1 \\ -1 & 5 & -1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & -1 & 5 & -1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & -1 & 5 & -1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 5 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \dots & 5 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \dots & -1 & 5 & -1 \\ -1 & 1 & 1 & 1 & 1 & \dots & 1 & -1 & 5 \end{array} \right) \quad \left(\begin{array}{ccccccccc} 5 & 1 & -1 & 1 & 1 & \dots & 1 & -1 & 1 \\ 1 & 5 & 1 & -1 & 1 & \dots & 1 & 1 & -1 \\ -1 & 1 & 5 & 1 & -1 & \dots & 1 & 1 & 1 \\ 1 & -1 & 1 & 5 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 5 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \dots & 5 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & \dots & 1 & 5 & 1 \\ 1 & -1 & 1 & 1 & 1 & \dots & -1 & 1 & 5 \end{array} \right) \\ n \geq 4 \text{ even} \qquad \qquad \qquad n \geq 3 \text{ odd} \end{array}$$

with $\det \alpha_2 = 2^{n-1}n^3$ for any $n \geq 2$.

Now suppose that $n = p$ is an odd prime, so

$$A \otimes V_1 = W_{L/K}(A)^2 \times W_{M/K}(A), \quad A \otimes V_2 = A^2 \times W_{F/K}(A).$$

By Corollary 4.5, it suffices to show that $\text{ord}_p Q(\phi_{f^t f})$ has the same parity as the left-hand side of the formula in the proposition. Since $f^t f = \alpha_1 \oplus \alpha_2 \oplus \alpha_3$, it remains to determine $\text{ord}_p Q(\alpha_i)$. Clearly $Q(\alpha_1)$ is prime to p . Next, α_3 acts as multiplication by 2 (resp. $4p$) on the trivial (resp. “sign”) component of $\mathbb{Q}[G/\langle g \rangle]$, so $\text{ord}_p Q(\alpha_3) = \text{rk}_p(A/M) - \text{rk}_p(A/K)$.

Finally, α_2 on $\mathbb{Q}[G/\langle g^{-2}h \rangle] \cong \mathbf{1} \oplus \rho$ has determinant p on $\mathbf{1}$ and therefore determinant $2^{p-1}p^2$ on ρ . As $\mathbf{1}, \rho$ are \mathbb{Q}_p -irreducible, Theorem 4.7 applies:

$$\text{ord}_p Q(\alpha_2) = \text{rk}_p(A, \mathbf{1}) + \frac{2}{p-1} \text{rk}_p(A, \rho).$$

Since $\text{rk}_p(A, \rho) = \text{rk}_p(A/L) - \text{rk}_p(A/K)$, this completes the proof. \square

Remark 4.18. Let E/K be an elliptic curve, and for simplicity let $p > 3$. Then

$$\text{ord}_p \frac{C(E/F)}{C(E/M)} \equiv |S_1| + |S_2| \pmod{2},$$

where S_1 (resp. S_2) is the set of primes v of M that ramify in F/M where E has split multiplicative reduction (resp. additive reduction, $v|p$, M_v/\mathbb{Q}_p has odd residue degree, and $\lfloor p \text{ord}_v(\Delta_v)/12 \rfloor$ is odd; Δ_v is the minimal discriminant of E at v). So if $\text{rk}_p(E/M) + |S_1| + |S_2|$ is odd, then

$$\text{rk}_p(E/L) \geq \text{rk}_p(E/K) + \frac{p-1}{2}.$$

4.6. Application to the p -Parity Conjecture over \mathbb{Q} .

Theorem 4.19 (=Theorem 1.4). *For every elliptic curve E/\mathbb{Q} and every prime p ,*

$$\text{rk}_p(E/\mathbb{Q}) \equiv \text{ord}_{s=1} L(E, s) \pmod{2}.$$

Proof. For $p = 2$ this is due to Monsky [26], so suppose p is odd. (Presumably the proof below would work for modular abelian varieties over totally real fields.)

By the results of Bump–Friedberg–Hoffstein–Murty–Murty–Waldspurger [4, 27, 37], there is an imaginary quadratic field M_0 where all bad primes of E split, and such that the quadratic twist of E by M_0 has analytic rank at most 1. By Kolyvagin’s theorem [20], the parity conjecture holds for the twist, so it suffices to prove it for E/M_0 .

Let M_n denote the n -th layer in the anticyclotomic \mathbb{Z}_p -extension of M_0 . The parity of the analytic rank is the same over M_n as over M_0 since the root number is unchanged in cyclic odd-degree extensions. The same holds for the p^∞ -Selmer rank (Corollary 4.15), so it suffices to prove the parity statement for E/M_n for some n .

Embedding M_{n+1} in \mathbb{C} , complex conjugation acts on the cyclic group $\text{Gal}(M_{n+1}/M_0)$ as -1 , so $\text{Gal}(M_{n+1}/\mathbb{Q})$ is dihedral. Write

$$F = M_{n+1}, \quad M = M_n, \quad L = M_{n+1} \cap \mathbb{R}, \quad K = M_n \cap \mathbb{R}.$$

Then $H = \text{Gal}(F/K) \cong D_{2p}$. It has three \mathbb{Q}_p -irreducible p -adic representations: trivial $\mathbf{1}$, sign ϵ and $(p-1)$ -dimensional ρ . As before, write $\mathcal{X}_p(E/k)$

for the dual Selmer group $\text{Hom}(\text{Sel}_{p^\infty}(E/k), \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and decompose

$$\mathcal{X} = \mathcal{X}_p(E/F) \cong \mathbf{1}^{\oplus m_1} \oplus \epsilon^{\oplus m_\epsilon} \oplus \rho^{\oplus m_\rho}.$$

As $\mathcal{X}_p(E/K) = \mathcal{X}^H$ etc. (Lemma 4.14),

$$\text{rk}_p(E/K) = m_1, \quad \text{rk}_p(E/M) = m_1 + m_\epsilon, \quad \text{rk}_p(E/L) = m_1 + \frac{p-1}{2}m_\rho.$$

Now we invoke Proposition 4.17:

$$\text{rk}_p(E/M) + m_\rho \equiv \text{ord}_p \frac{C(E/F)}{C(E/M)} \pmod{2}.$$

Since all bad primes of E split in M/K , the root number $w(E/M) = -1$ and both $C(E/F)$ and $C(E/M)$ are squares. So the right-hand side in the above formula is zero, and it suffices to show that m_ρ is odd.

Now take n large enough. Then Cornut–Vatsal’s [9] Thm. 1.5 provides a primitive character χ of $\text{Gal}(F/M_0)$ such that the twisted L -function $L(E/M_0, \chi, s)$ has a simple zero at $s = 1$. Their theorem requires N_E, Δ_{M_0} and p to be coprime, but as they explain this is only necessary to invoke the Gross–Zagier–Zhang formula; this formula has now proved in complete generality by Yuan–Zhang–Zhang [39].

By Tian–Zhang [36], which generalises the earlier work by Bertolini–Darmon [1], the χ -component of \mathcal{X} has multiplicity 1. So \mathcal{X} contains exactly one copy of the unique $(p-1)p^n$ -dimensional \mathbb{Q}_p -irreducible p -adic representation of $\text{Gal}(F/\mathbb{Q})$. Its restriction to H is $\rho^{\oplus p^n}$, and no other representation contributes to ρ , so $m_\rho = p^n$ is odd.

(As an alternative to the yet unavailable [39, 36], one may bypass the L -functions completely by combining Cornut–Vatsal’s [9] Thm. 4.2 with Nekovář’s [29] Thm. 3.2. This directly yields a χ such that the χ -component of \mathcal{X} has multiplicity 1.) \square

Corollary 4.20. *For every E/\mathbb{Q} , either the Birch–Swinnerton-Dyer rank formula holds modulo 2 (Conjecture 1.1), or $\text{III}(E/\mathbb{Q})$ contains a copy of \mathbb{Q}/\mathbb{Z} .*

REFERENCES

- [1] M. Bertolini, H. Darmon, Iwasawa’s Main Conjecture for elliptic curves over anticyclotomic \mathbb{Z}_p -extensions, Annals of Math. 162, Number 1 (2005), 1–64.
- [2] B. J. Birch, Conjectures concerning elliptic curves, Proc. Sympos. Pure Math., Vol. VIII (1965), Amer. Math. Soc., Providence, R.I, 106–112.
- [3] B. J. Birch, N. M. Stephens, The parity of the rank of the Mordell-Weil group, Topology 5 (1966), 295–299.
- [4] D. Bump, S. Friedberg, and J. Hoffstein, Nonvanishing theorems for L-functions of modular forms and their derivatives, Invent. Math. 102 (1990), 543–618.
- [5] J. W. S. Cassels, Arithmetic on curves of genus 1, IV, Proof of the Hauptvermutung, J. Reine Angew. Math. 211, (1962), 95–112.
- [6] J. W. S. Cassels, Arithmetic on curves of genus 1. VIII: On conjectures of Birch and Swinnerton-Dyer, J. Reine Angew. Math. 217 (1965), 180–199 (1965).
- [7] J. Coates, T. Fukaya, K. Kato, R. Sujatha, Root numbers, Selmer groups and non-commutative Iwasawa theory, preprint.
- [8] J. Coates, R. Sujatha, appendix to T. Dokchitser, V. Dokchitser, Computations in non-commutative Iwasawa theory, Proc. London Math. Soc. (3) 94 (2006) 211–272.

- [9] C. Cornut, V. Vatsal, Nontriviality of Rankin-Selberg L-functions and CM points, in: L-functions and Galois representations (Durham, July 2004), LMS Lecture Note Series 320 (2007), Cambridge Univ. Press, 121–186.
- [10] C. Diem, N. Naumann, On the structure of Weil restrictions of Abelian varieties, J. Ramanujan Math. Soc. 18, No.2 (2003), 153–174.
- [11] T. Dokchitser, V. Dokchitser, Parity of ranks for elliptic curves with a cyclic isogeny, J. Number Theory 128 (2008), 662–679.
- [12] T. Dokchitser, V. Dokchitser, Self-duality of Selmer groups, 2007, arxiv: 0705.1899, to appear in Math. Proc. Cam. Phil. Soc.
- [13] T. Dokchitser, V. Dokchitser, Regulator constants and the parity conjecture, 2007, arxiv: 0709.2852.
- [14] T. Fisher, Appendix to V. Dokchitser, Root numbers of non-abelian twists of elliptic curves, Proc. London Math. Soc. (3) 91 (2005), 300–324.
- [15] R. Greenberg, On the Birch and Swinnerton-Dyer conjecture, Invent. Math. 72, no. 2 (1983), 241–265.
- [16] L. Guo, General Selmer groups and critical values of Hecke L-functions, Math. Ann. 297 no. 2 (1993), 221–233.
- [17] Y. Hachimori, O. Venjakob, Completely faithful Selmer groups over Kummer extensions, Documenta Mathematica, Extra Volume: Kazuya Kato's Fiftieth Birthday (2003), 443–478.
- [18] B. D. Kim, The Parity Theorem of Elliptic Curves at Primes with Supersingular Reduction, Compositio Math. 143 (2007) 47–72.
- [19] S. Kobayashi, The local root number of elliptic curves with wild ramification, Math. Ann. 323 (2002), 609–623.
- [20] V. A. Kolyvagin, Euler systems, The Grothendieck Festschrift, Prog. in Math., Boston, Birkhauser (1990).
- [21] K. Kramer, Arithmetic of elliptic curves upon quadratic extension, Trans. Amer. Math. Soc. 264 (1981), 121–135.
- [22] K. Kramer, J. Tunnell, Elliptic curves and local ϵ -factors, Compositio Math. 46 (1982), 307–352.
- [23] B. Mazur, K. Rubin, Finding large Selmer ranks via an arithmetic theory of local constants, Annals of Math. 166 (2), 2007, 579–612.
- [24] J. S. Milne, On the arithmetic of abelian varieties, Invent. Math. 17 (1972), 177–190.
- [25] J. S. Milne, Arithmetic duality theorems, Perspectives in Mathematics, No. 1, Academic Press, 1986.
- [26] P. Monsky, Generalizing the Birch–Stephens theorem. I: Modular curves, Math. Z., 221 (1996), 415–420.
- [27] M. R. Murty and V. K. Murty, Mean values of derivatives of modular L-series, Annals of Math. 133 (1991), 447–475.
- [28] J. Nekovář, Selmer complexes, Astérisque 310 (2006).
- [29] J. Nekovář, The Euler system method for CM points on Shimura curves, in: L-functions and Galois representations (Durham, July 2004), LMS Lecture Note Series 320 (2007), Cambridge Univ. Press, 471–547.
- [30] B. Poonen, M. Stoll, The Cassels–Tate pairing on polarized abelian varieties, Annals of Math. 150 (1999), 1109–1149.
- [31] D. E. Rohrlich, Variation of the root number in families of elliptic curves, Compositio Math. 87 (1993), 119–151.
- [32] J.-P. Serre, Linear Representations of Finite Groups, GTM 42, Springer Verlag 1977.
- [33] J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, GTM 151, Springer-Verlag 1994.
- [34] J. Tate, Duality theorems in Galois cohomology over number fields, Proc. ICM Stockholm 1962, 234–241.

- [35] J. Tate, On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, Séminaire Bourbaki, 18e année, 1965/66, no. 306.
- [36] Y. Tian, S. Zhang, Euler system of CM-points on Shimura curves, in preparation.
- [37] J.-L. Waldspurger, Correspondances de Shimura et quaternions, Forum Math. 3 (1991), 219–307.
- [38] H. Yu, Idempotent relations and the conjecture of Birch and Swinnerton-Dyer, Math. Ann. 327 (2003), 67–78.
- [39] H. Yuan, S. Zhang, W. Zhang, Heights of CM points I: Gross-Zagier formula, 2008, preprint.

ROBINSON COLLEGE, CAMBRIDGE CB3 9AN, UNITED KINGDOM
E-mail address: t.dokchitser@dpmms.cam.ac.uk

GONVILLE & CAIUS COLLEGE, CAMBRIDGE CB2 1TA, UNITED KINGDOM
E-mail address: v.dokchitser@dpmms.cam.ac.uk